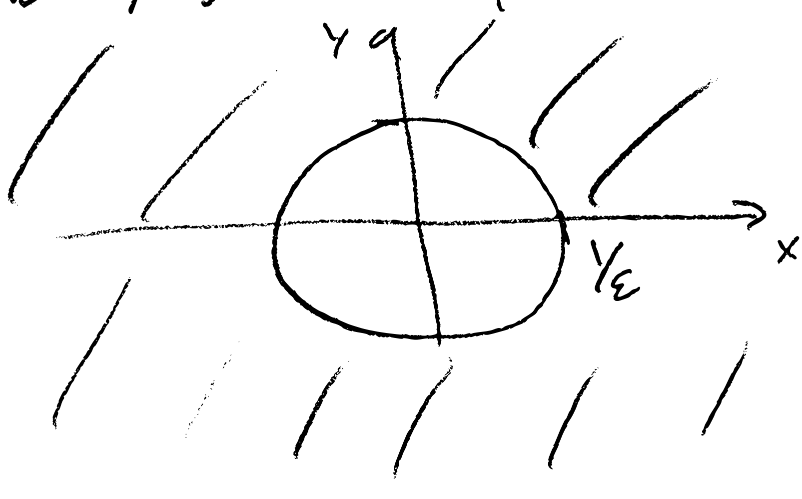
Extended plane $\mathbb{C}_\infty$ / Riemann sphere / Cplx projective plane $\mathbb{CP}^1$.

Obs. The complex plane $\mathbb{C}$ is not compact. $\exists \{z_n\}$ s.t. does not converge. In $\mathbb{R}$ one often adds 2 pts $\pm\infty$ to deal with this. In $\mathbb{C}$ we add 1 pt ∞.

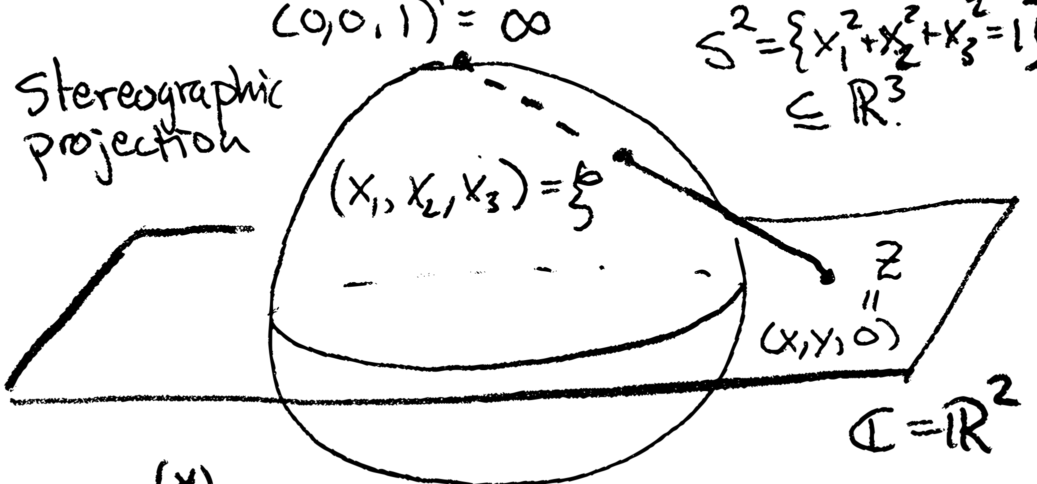
① As a set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Def. An open disks centered at ∞:

$$B(\infty, \varepsilon) = \{\infty\} \cup \{z \in \mathbb{C} : |z| \geq 1/\varepsilon\}$$



② For geometry, identify $\mathbb{C}_\infty$ w/ Riemann sphere.



(*)

$$\xi = t(0,0,1) + (1-t)(x,y,0) \quad \text{s.t. } |\xi|^2 = 1.$$

Solve for t in $|\xi|^2 = 1 \Rightarrow$

$$t = \frac{|z|^2 - 1}{|z|^2 + 1} \qquad (|z|^2 = x^2 + y^2).$$

$\Rightarrow$

$$x_1 = \frac{2x}{|z|^2 + 1}, \quad x_2 = \frac{2y}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Conversely:

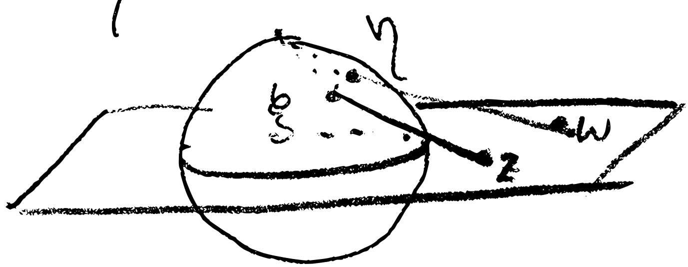
(*) I think I switched the roles of t and $(1-t)$ in class. Formulas for t are for this order.

Given $\xi = (x_1, x_2, x_3) \neq \infty$ on S^2, set

$\underline{t = x_3} \Rightarrow$

$$z = \frac{x_1 + i x_2}{1 - x_3}.$$

Geometry on $\mathbb{C}_\infty$.



For $z, w \in \mathbb{C}$ (not ∞), let

$d_\infty(z,w) = |\xi - \eta|$ (in $\mathbb{R}^3$) $\Rightarrow$

$$\left\{ \begin{array}{l} d_\infty(z,w) = \dfrac{2|z-w|}{\left[(1+|z|^2)(1+|w|^2)\right]^{1/2}} \\ \quad \text{and} \\ d_\infty(z,\infty) = \dfrac{2}{(1+|z|^2)^{1/2}} \end{array} \right.$$

Fubini-Study metric

to be discussed.

Prop!

$$B(\infty,\varepsilon)\setminus\{\infty\} = \{z : d_\infty(z,\infty) < \delta\}$$

where $\varepsilon = \dfrac{\delta}{\sqrt{4-\delta^2}}$ $\quad(\delta < 2)$.

Pf. $z \in B(\infty,\varepsilon)\setminus\{\infty\} \iff \boxed{|z| > 1/\varepsilon}$.

Now $d_\infty(z,\infty) < \delta \iff$

$$\frac{2}{(1+|z|^2)^{1/2}} < \delta$$

or $\quad 1+|z|^2 > \dfrac{4}{\delta^2} \Rightarrow \boxed{|z|^2 > \dfrac{4}{\delta^2} - 1}$.

$$\frac{1}{\varepsilon^2} = \frac{4}{\delta^2} - 1 \quad\text{or}\quad \varepsilon = \frac{\delta}{\sqrt{4-\delta^2}}. \quad \blacksquare$$